Capacity and Optimal Collusion Attack Channels for Gaussian Fingerprinting Games

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1. INTRODUCTION

In content fingerprinting, the same media covertext—image, video, audio, or text—is distributed to many users. A fingerprint, a mark unique to each user, is embedded into each copy of the distributed covertext. In a collusion attack, two or more users may combine their copies in an attempt to “remove” their fingerprints and forge a pirated copy. To trace the forgery back to members of the coalition, we need fingerprinting codes that can reliably identify the fingerprints of those members. Researchers have been focusing on designing or testing fingerprints for Gaussian host signals and the mean square error (MSE) distortion under some classes of collusion attacks, in terms of the detector’s error probability in detecting collusion members. For example, under the assumptions of Gaussian fingerprints and Gaussian attacks (the fingerprinted signals are averaged and then the result is passed through a Gaussian test channel), Moulin and Briassouli¹ derived optimal strategies in a game-theoretic framework that uses the detector’s error probability as the performance measure for a binary decision problem (whether a user participates in the collusion attack or not); Stone² and Zhao *et al.*³ studied average and other non-linear collusion attacks for Gaussian-like fingerprints; Wang *et al.*⁴ stated that the average collusion attack is the most efficient one for orthogonal fingerprints; Kiyavash and Moulin⁵ derived a mathematical proof of the optimality of the average collusion attack under some assumptions.

In this paper, we also consider Gaussian cover signals, the MSE distortion, and memoryless collusion attacks. We do not make any assumption about the fingerprinting codes used other than an embedding distortion constraint. Also, our only assumptions about the attack channel are an expected distortion constraint, a memoryless constraint, and a fairness constraint. That is, the colluders are allowed to use any *arbitrary nonlinear strategy* subject to the above constraints. Under those constraints on the fingerprint embedder and the colluders, fingerprinting capacity is obtained as the solution of a mutual-information game involving probability density functions (pdf’s) designed by the embedder and the colluders. We show that the *optimal fingerprinting strategy* is a Gaussian test channel where the fingerprinted signal is the sum of an attenuated version of the cover signal plus a Gaussian information-bearing noise, and the *optimal collusion strategy* is to average fingerprinted signals possessed by all the colluders and pass the averaged copy through a Gaussian test channel. *The capacity result and the optimal strategies are the same for both the private and public games.* In the former scenario, the original covertext is available to the decoder, while in the latter setup, the original covertext is available to the encoder but not to the decoder.

2. STATEMENT OF THE PROBLEM

Both of the private and public fingerprinting setups are modelled in Fig. 1. Let the covertext sequence $\mathbf{S} = (S_1, \dots, S_N)$ consist of independent and identically distributed (i.i.d.) samples drawn from a Gaussian pdf $\mathcal{N}(0, \sigma^2)$. The fingerprinting encoder generates $|\mathcal{M}| = \lceil 2^{NR} \rceil$ fingerprinted copies $\mathbf{X}_m = f_N(\mathbf{S}, m)$, where $f_N : \mathbb{R}^N \times \mathcal{M} \rightarrow \mathbb{R}^N$ is the fingerprinting code and R is its rate. Each fingerprint m corresponds to a user. Assume that the collusion size is $L = 2$.^{*} Let M_1 and M_2 be two users selected independently and uniformly from the set $\mathcal{M}$. They collude to produce a pirated copy, or forgery, $\mathbf{Y} \in \mathbb{R}^N$, through an attack channel $p_{\mathbf{Y}|\mathbf{X}_1\mathbf{X}_2}$. The decoder does not know $p_{\mathbf{Y}|\mathbf{X}_1\mathbf{X}_2}$ selected by the two colluders. The decoder produces estimates $(\hat{M}_1, \hat{M}_2) = g_N(\mathbf{Y}) \in \mathcal{M}^2$ in attempt to catch the two colluders.

^{*}In this paper, we focus on the case of two colluders for the simplicity of notation. Although cumbersome, it is conceptually straightforward to extend to the case where there are more than two colluders.

Figure 1. Fingerprinting games.

2.1. Notation

We use uppercase letters for random variables, lowercase letters for their individual values, and boldface fonts for sequences (or vectors). Differential entropy of a random variable X is denoted by $h(X)$, and the mutual information between two random variables X and Y is denoted by $I(X; Y)$. The mathematical expectation is denoted by $\mathbb{E}$.

2.2. Constrained Fingerprint Codes and Collusion Attack Channels

The MSE distortion between a fingerprinted signal sequence $\mathbf{x}$ and a covert text signal sequence $\mathbf{s}$ is defined as

$$d^N(\mathbf{s}, \mathbf{x}) = \frac{1}{N} \sum_{i=1}^N (x_i - s_i)^2. \quad (1)$$

Fingerprinting codes are required to satisfy

$$\mathbb{E} d^N(\mathbf{S}, \mathbf{X}) \leq D_1,$$

where $D_1 \geq 0$.

DEFINITION 2.1. A covert private fingerprinting channel is a conditional pdf $p_{X|S}(x|s)$ such that

$$\sum_{s,x} p_{X|S}(x|s) p_S(s) (x - s)^2 \leq D_1. \quad (2)$$

We denote by $\mathcal{Q}_1$ the set of such feasible covert private fingerprinting channels.

DEFINITION 2.2. Given an auxiliary random variable $U \in \mathbb{R}$, a covert public fingerprinting channel is a conditional pdf $p_{XU|S}(x, u|s)$ such that

$$\sum_{u,s,x} p_{XU|S}(x, u|s) p_S(s) (x - s)^2 \leq D_1. \quad (3)$$

We denote by $\mathcal{Q}_2$ the set of such feasible covert public fingerprinting channels.

We define a constrained discrete memoryless collusion channel (DMCC) as a pdf $p_{Y|X_1 X_2}$ subject to a fairness constraint

$$p_{Y|X_1 X_2} = p_{Y|X_2 X_1} \quad (4)$$

and the expected MSE distortion constraint

$$\mathbb{E}d^N(\mathbf{X}, \mathbf{Y}) \leq D_2. \quad (5)$$

We pose the fairness constraint and the D_2 constraint on both colluders, based on the assumption that neither colluder is willing to take more risks than the other colluder. If this assumption does not hold, then any attempt to catch all colluders is futile because one colluder can make negligible or even zero contribution to the pirated copy $\mathbf{Y}$.

DEFINITION 2.3. A discrete memoryless collusion attack channel is a conditional pdf $p_{Y|X_1X_2}(y|x_1, x_2)$ such that

$$p_{Y|X_1X_2}(y|x_1, x_2) = p_{Y|X_2X_1}(y|x_2, x_1),$$

and

$$\sum_{x_1, x_2, y} p_{X_1}(x_1)p_{X_2}(x_2)p_{Y|X_1X_2}(y|x_1, x_2)(y - x_i)^2 \leq D_2, \text{ for } i = 1, 2. \quad (6)$$

We denote $\mathcal{A}$ the set of all such feasible DMMCs.

2.3. Probability of Error and Capacity

In this paper, we only consider the case of detecting at least one colluder. An error occurs when the decoded two fingerprints do not match the fingerprints of the real colluders:

$$\hat{M}_1 \notin \{M_1, M_2\} \text{ and } \hat{M}_2 \notin \{M_1, M_2\}.$$

The average probability of error for a deterministic code (f_N, g_N) when a DMCC $p_{Y|X_1X_2}$ is in effect is denoted by $P_e(f_N, g_N, p_{Y|X_1X_2})$. The minmax probability of error for the class of constrained DMCCs considered is given by

$$P_{e,N} = \min_{(f_N, g_N)} \max_{p_{Y|X_1X_2}} P_e(f_N, g_N, p_{Y|X_1X_2}). \quad (7)$$

DEFINITION 2.4. A fingerprinting rate R is achievable if there exists a code (f_N, g_N) such that $|\mathcal{M}| = \lceil 2^{NR} \rceil$ and $P_{e,N} \rightarrow 0$ as $N \rightarrow \infty$.

DEFINITION 2.5. The capacity $C(D_1, D_2)$ is the supremum of all achievable rates.

2.4. Known Results on Capacity Formulas

Somekh-Baruch and Merhav^{6,7} derived capacity and error exponent formulas for private fingerprinting, and Wang and Moulin⁸ derived capacity and error exponent formulas for public fingerprinting. These results⁶⁻⁸ were derived for the case that the cover, fingerprinted, and forgery signals are from finite alphabets. But the results can be extended to the case of continuous alphabets. We use the following known formulas and derive the capacity for the case of a Gaussian cover signal and the MSE distortion. Capacity is infinite if the feasible set of attacks $\mathcal{A}$ is empty.

THEOREM 2.6.⁶ The capacity of the **private** fingerprinting game is given by

$$C(D_1, D_2) = \max_{p_{X|S} \in \mathcal{Q}_1} \min_{p_{Y|X_1X_2} \in \mathcal{A}} \frac{1}{2} I(X_1, X_2; Y|S), \quad (8)$$

where X_1 is independent of X_2 given S , $p_{X|S} = p_{X_1|S} = p_{X_2|S}$, and $S \rightarrow (X_1, X_2) \rightarrow Y$ forms a Markov chain. That is, the joint pdf of (S, X_1, X_2, Y) is given by

$$p_{SX_1X_2Y}(s, x_1, x_2, y) = p_S(s)p_{X|S}(x_1|s)p_{X|S}(x_2|s)p_{Y|X_1X_2}(y|x_1, x_2).$$

THEOREM 2.7.⁸ The capacity of the **public** fingerprinting game is given by

$$C(D_1, D_2) = \max_{p_{XU|S} \in \mathcal{Q}_2} \min_{p_{Y|X_1X_2} \in \mathcal{A}} \frac{1}{2} [I(U_1, U_2; Y) - I(U_1, U_2; S)], \quad (9)$$

where $U_1 \in \mathbb{R}$ and $U_2 \in \mathbb{R}$ are auxiliary random variables, (X_1, U_1) is independent of (X_2, U_2) given S , $p_{XU|S} = p_{X_1U_1|S} = p_{X_2U_2|S}$, and $(S, U_1, U_2) \rightarrow (X_1, X_2) \rightarrow Y$ forms a Markov chain. That is, the joint pdf of $(S, U_1, U_2, X_1, X_2, Y)$ is given by

$$p_{SU_1U_2X_1X_2Y}(s, u_1, u_2, x_1, x_2, y) = p_S(s)p_{XU|S}(x_1, u_1|s)p_{XU|S}(x_2, u_2|s)p_{Y|X_1X_2}(y|x_1, x_2).$$

3. MAIN RESULTS

THEOREM 3.1. For a Gaussian cover signal $S \sim \mathcal{N}(0, \sigma^2)$, the **private** Gaussian fingerprinting game capacity $C^{\text{Priv}}(D_1, D_2)$ derived from (8) is determined as follows:

1. $C^{\text{Priv}}(D_1, D_2) = 0$ if $D_2 \geq (\sqrt{D_1} + \sigma)^2$;
2. $C^{\text{Priv}}(D_1, D_2) = \infty$ if $D_2 < \frac{1}{2}D_1$; and
3. if $\frac{1}{2}D_1 \leq D_2 < (\sqrt{D_1} + \sigma)^2$,

$$C^{\text{Priv}}(D_1, D_2) = \max_{\tilde{a} \in \left[1 - \sqrt{\frac{D_1}{\sigma^2}}, 1 + \sqrt{\frac{D_1}{\sigma^2}}\right]} \frac{1}{4} \log \left[1 + \frac{2\sigma_Z^2(\tilde{a}^2\sigma^2 + \sigma_Z^2 - D_2)}{(2\tilde{a}^2\sigma^2 + \sigma_Z^2)(2D_2 - \sigma_Z^2)} \right], \quad (10)$$

where

$$\sigma_Z^2 = D_1 - (\tilde{a} - 1)^2\sigma^2.$$

Denote the maximizing $\tilde{a}$ of (10) by a . The maxmin in (8) is achieved by the following fingerprinting channel:

$$X_1 = aS + Z_1, \quad X_2 = aS + Z_2,$$

where $Z_1 \sim \mathcal{N}(0, \sigma_Z^2)$ and $Z_2 \sim \mathcal{N}(0, \sigma_Z^2)$ are independent of S ; the optimal collusion attack channel is

$$Y = b(X_1 + X_2 + W),$$

where $W \sim \mathcal{N}(0, \sigma_W^2)$ is independent of X_1 and X_2 ,

$$b = \frac{a^2\sigma^2 + \sigma_Z^2 - D_2}{2a^2\sigma^2 + \sigma_Z^2},$$

and

$$\sigma_W^2 = \frac{(2a^2\sigma^2 + \sigma_Z^2)(2D_2 - \sigma_Z^2)}{a^2\sigma^2 + \sigma_Z^2 - D_2}.$$

THEOREM 3.2. For a Gaussian cover signal $S \sim \mathcal{N}(0, \sigma^2)$, the **public** Gaussian fingerprinting game capacity derived from (9) is the same as the private game capacity:

$$C^{\text{Pub}} = C^{\text{Priv}}.$$

When $\frac{1}{2}D_1 \leq D_2 < (\sqrt{D_1} + \sigma)^2$, the optimal fingerprinting channel is

$$X_1 = aS + Z_1, \quad X_2 = aS + Z_2,$$

and

$$U_1 = \alpha S + Z_1, \quad U_2 = \alpha S + Z_2,$$

where $Z_1 \sim \mathcal{N}(0, \sigma_Z^2)$ and $Z_2 \sim \mathcal{N}(0, \sigma_Z^2)$ are independent of S , a is the maximizer of (10),

$$\sigma_Z^2 = D_1 - (a - 1)^2 \sigma^2,$$

and

$$\alpha = \frac{2a \sigma_Z^2}{2\sigma_Z^2 + \sigma_W^2};$$

the optimal collusion attack channel is

$$Y = b(X_1 + X_2 + W),$$

where $W \sim \mathcal{N}(0, \sigma_W^2)$ is independent of X_1 and X_2 ,

$$b = \frac{a^2 \sigma^2 + \sigma_Z^2 - D_2}{2a^2 \sigma^2 + \sigma_Z^2},$$

and

$$\sigma_W^2 = \frac{(2a^2 \sigma^2 + \sigma_Z^2)(2D_2 - \sigma_Z^2)}{a^2 \sigma^2 + \sigma_Z^2 - D_2}.$$

Remark 1: For the Gaussian cover signal S and the MSE distortion, the achievable rate of reliable transmission (the decoder correctly decodes at least one colluder's fingerprint) is the same whether or not the cover signal is available to the decoder! The same observation has been made for the watermarking case,⁹ which is a single-user version of the fingerprinting game.

Remark 2: In the small distortion case, $\sigma^2 \gg D_1, D_2$, when $\frac{1}{2}D_1 \leq D_2 < (\sqrt{D_1} + \sigma)^2$, we have the optimal parameters: $a^* \approx 1$, $\sigma_Z^{2*} \approx D_1$, $b^* \approx \frac{1}{2}$, $\sigma_W^{2*} \approx 2(2D_2 - D_1)$ and

$$C^{Priv} = C^{Pub} \approx \frac{1}{4} \log \left[1 + \frac{D_1}{2D_2 - D_1} \right].$$

Remark 3: Capacity is infinite if $D_2 < \frac{1}{2}D_1$ because the feasible set $\mathcal{A}$ is empty in that case.

4. SKETCH OF THE PROOF OF THEOREM 3.1

The maximal value of $\mathbb{E}X^2$ over all feasible $p_{X|S}$ is $(\sqrt{D_1} + \sigma)^2$ and achieved by $X = aS$ when $a = 1 + \sqrt{\frac{D_1}{\sigma^2}}$. The colluders can simply let $Y = 0$ if $D_2 \geq \mathbb{E}X^2$, and the value of the cost function of (8) is 0. Thus, the capacity C^{Priv} is 0 if $D_2 \geq (\sqrt{D_1} + \sigma)^2$.

The rest of Theorem 3.1, i.e. when $D_2 < (\sqrt{D_1} + \sigma)^2$, is proved by showing that the capacity C^{Priv} is lower bounded and at the same time upper bounded by the value given in the theorem.

4.1. Proof of the Direct Part: Lower Bound of C^{Priv}

We obtain a lower bound on C^{Priv} by fixing a (not necessarily optimal) covert private fingerprinting channel and looking for the minimizing collusion attack channel for this specific fingerprinting channel.

Suppose that $X_1 = aS + Z_1$ and $X_2 = aS + Z_2$, where the cover signal S and fingerprints $Z_1 \sim \mathcal{N}(0, \sigma_Z^2)$, $Z_2 \sim \mathcal{N}(0, \sigma_Z^2)$ are mutually independent, and

$$\sigma_Z^2 = D_1 - (a - 1)^2 \sigma^2. \tag{11}$$

Only when

$$a \in \left[1 - \sqrt{\frac{D_1}{\sigma^2}}, 1 + \sqrt{\frac{D_1}{\sigma^2}} \right], \tag{12}$$

the variance σ_Z^2 is greater than 0 and fingerprints are nonzero.

Thus,

$$\mathbb{E}(X_1 - S)^2 = \mathbb{E}(X_2 - S)^2 = (a - 1)^2 \sigma^2 + \sigma_Z^2 = D_1, \quad (13)$$

and the assumed covert private fingerprinting channel is feasible.

We have

$$\mathbb{E}X_1X_2 = a^2\sigma^2. \quad (14)$$

and

$$\mathbb{E}X^2 = \mathbb{E}X_{1,2}^2 = a^2\sigma^2 + \sigma_Z^2 = D_1 + (2a - 1)\sigma^2. \quad (15)$$

In order for $\mathbb{E}X^2 \geq 0$, i.e. $X_{1,2} \neq 0$, it is required that

$$a \geq \frac{1}{2} \left(1 - \frac{D_1}{\sigma^2} \right). \quad (16)$$

Since $1 - \sqrt{x} \geq \frac{1}{2}(1 - x)$ for $x \geq 0$, the interval given by (12) suffices to specify the meaningful values of a and (16) does not impose an additional constraint.

From the fairness condition $p_{Y|X_1X_2} = p_{Y|X_2X_1}$, we have $\mathbb{E}XY = \mathbb{E}X_1Y = \mathbb{E}X_2Y$. Let

$$b = \frac{\mathbb{E}XY}{\mathbb{E}X^2 + \mathbb{E}X_1X_2},$$

$$V = b^{-1}Y,$$

and

$$W = V - X_1 - X_2.$$

Thus

$$Y = b(X_1 + X_2 + W),$$

$$\mathbb{E}XW = \mathbb{E}X_1W = \mathbb{E}X_2W = 0,$$

and a feasible DMCC needs to satisfy

$$\mathbb{E}(Y - X_1)^2 = \mathbb{E}(Y - X_2)^2 \leq D_2. \quad (17)$$

Let us denote $\sigma_W^2 = \mathbb{E}W^2$, then from (17)

$$\begin{aligned} \sigma_W^2 &\leq b^{-2} [D_2 - (b - 1)^2 \mathbb{E}X^2 - b^2 \mathbb{E}X^2 - 2b(b - 1)a^2\sigma^2] \\ &= b^{-2} [D_2 - \mathbb{E}X^2 - 2b(b - 1)(\mathbb{E}X^2 + a^2\sigma^2)]. \end{aligned} \quad (18)$$

We substitute all the above-defined random variables to the cost function of (8):

$$\begin{aligned} J = \frac{1}{2} I(X_1, X_2; Y|S) &\stackrel{(a)}{=} \frac{1}{2} I(X_1, X_2; V|S) \\ &= \frac{1}{2} I(aS + Z_1, aS + Z_2; aS + Z_1 + aS + Z_2 + W|S) \\ &\stackrel{(b)}{=} \frac{1}{2} I(Z_1, Z_2; Z_1 + Z_2 + W|S) \\ &= \frac{1}{2} [h(Z_1, Z_2|S) - h(Z_1 + Z_2 + W|S, Z_1, Z_2)] \\ &\stackrel{(c)}{\geq} \frac{1}{2} [h(Z_1, Z_2) - h(Z_1 + Z_2 + W|Z_1, Z_2)] \\ &= \frac{1}{2} I(Z_1, Z_2; Z_1 + Z_2 + W) \\ &\stackrel{(d)}{\geq} \frac{1}{2} I(Z_1, Z_2; Z_1 + Z_2 + W^*) \\ &= \frac{1}{2} [h(Z_1 + Z_2 + W^*) - h(W^*)] \\ &= \frac{1}{4} \log \left[1 + \frac{2\sigma_Z^2}{\sigma_W^2} \right], \end{aligned} \quad (19)$$

where (a) follows from the fact that $Y = bV$ is a deterministic function of V ; (b) is valid because the mutual information is conditioned on S ; (c) follows from the facts that Z_1 and Z_2 are independent of S and conditioning decreases the entropy; (d) is obtained because a Gaussian independent noise $W^* \sim \mathcal{N}(0, \sigma_W^2)$ minimizes the mutual information when the noise power is fixed at σ_W^2 .¹⁰

We substitute (11) and (18) into (19). Given a , the colluders search for the optimal b and σ_W^2 to minimize the lower bound on J . Calculation shows that the optimal b is

$$b^* = \frac{a^2\sigma^2 + \sigma_Z^2 - D_2}{2a^2\sigma^2 + \sigma_Z^2}, \quad (20)$$

and the optimal σ_W^2 is

$$\sigma_W^{2*} = \frac{(2a^2\sigma^2 + \sigma_Z^2)(2D_2 - \sigma_Z^2)}{a^2\sigma^2 + \sigma_Z^2 - D_2}. \quad (21)$$

Then we seek a that minimizes (21) and therefore maximizes (19). Finally, the capacity C^{Priv} is lower-bounded by

$$C_G \triangleq \max_{a \in \left[1 - \sqrt{\frac{D_1}{\sigma^2}}, 1 + \sqrt{\frac{D_1}{\sigma^2}}\right]} \frac{1}{4} \log \left[1 + \frac{2\sigma_Z^2(a^2\sigma^2 + \sigma_Z^2 - D_2)}{(2a^2\sigma^2 + \sigma_Z^2)(2D_2 - \sigma_Z^2)} \right]. \quad (22)$$

Note that if $D_2 < \frac{1}{2}D_1$, then when $a = 1$, $\sigma_Z^2 = D_1$ and $\sigma_W^2 < 0$, i.e., the set of feasible DMCCs is empty and hence the capacity is infinite.

4.2. Proof of the Converse Part: $C^{Priv} \leq C_G$

We now only need to consider $\frac{1}{2}D_1 \leq D_2 < (\sqrt{D_1} + \sigma)^2$. We obtain an upper bound on C^{Priv} by fixing a (not necessarily optimal) collusion attack channel and looking for the maximizing fingerprinting channel for this specific collusion channel.

Let the collusion attack channel be $Y = bV = b(X_1 + X_2 + W)$, where $W \sim \mathcal{N}(0, \sigma_W^2)$ is independent of X_1 and X_2 , and b and σ_W^2 will be optimized subject to the distortion constraints:

$$\mathbb{E}(Y - X_1)^2 = (b - 1)^2 \mathbb{E}X_1^2 + b^2 \mathbb{E}X_2^2 + 2b(b - 1) \mathbb{E}X_1X_2 + b^2 \sigma_W^2 = D_2, \quad (23)$$

and

$$\mathbb{E}(Y - X_2)^2 = (b - 1)^2 \mathbb{E}X_2^2 + b^2 \mathbb{E}X_1^2 + 2b(b - 1) \mathbb{E}X_1X_2 + b^2 \sigma_W^2 = D_2. \quad (24)$$

Let $Z_1 = X_1 - aS$ and $Z_2 = X_2 - aS$, where Z_1 and Z_2 have the same pdf since $p_{X_1|S} = p_{X_2|S}$, and a is a parameter to be optimized. We denote $\sigma_Z^2 = \mathbb{E}Z_1^2 = \mathbb{E}Z_2^2$. A feasible covert private fingerprinting channel satisfies

$$\mathbb{E}(X_i - S)^2 = (a - 1)^2 \sigma^2 + \sigma_Z^2 + 2(a - 1) \mathbb{E}SZ_i \leq D_1,$$

for $i = 1, 2$.

Substituting all the above-defined random variables to the cost function of (8), we have

$$\begin{aligned} J &= \frac{1}{2} I(X_1, X_2; Y|S) \stackrel{(a)}{=} \frac{1}{2} I(X_1, X_2; V|S) \\ &= \frac{1}{2} I(X_1, X_2; X_1 + X_2 + W|S) \\ &= \frac{1}{2} [h(X_1 + X_2 + W|S) - h(X_1 + X_2 + W|S, X_1, X_2)] \\ &\stackrel{(b)}{=} \frac{1}{2} [h(X_1 + X_2 + W|S) - h(W)] \\ &= \frac{1}{2} [h(Z_1 + Z_2 + W|S) - h(W)] \\ &\stackrel{(c)}{\leq} \frac{1}{2} [h(Z_1 + Z_2 + W) - h(W)] \end{aligned}$$

$$\begin{aligned}
&\stackrel{(d)}{\leq} \frac{1}{2} [h(Z_1^* + Z_2^* + W) - h(W)] \\
&= \frac{1}{4} \log \left[1 + \frac{2\sigma_Z^2}{\sigma_W^2} \right], \tag{25}
\end{aligned}$$

where (a) follows from the fact that $Y = bV$ is a deterministic function of V ; (b) is valid because W is independent of S , X_1 , and X_2 ; (c) follows from the fact that conditioning decreases the entropy; (d) is obtained because Gaussian independent random variables $Z_1^* \sim \mathcal{N}(0, \sigma_Z^2)$ and $Z_2^* \sim \mathcal{N}(0, \sigma_Z^2)$ maximizes the entropy.¹⁰

Simple calculation shows that since $\frac{1}{2}D_1 \leq D_2 < (\sqrt{D_1} + \sigma)^2$, then $\sigma_W^2 \geq 0$ and $\sigma_Z^2 > 0$: the optimal b is the same as shown in (20), the optimal σ_W^2 is the same as shown in (21), the optimal σ_Z^2 is given by $\sigma_Z^2 = D_1 - (a-1)^2\sigma^2$ as in (11), the optimal a is the maximizer for (22), and finally, C^{Priv} is upper-bounded by C_G shown in (22).

5. SKETCH OF THE PROOF OF THEOREM 3.2

Since for the public Gaussian fingerprinting game, the cover signal S is not available to the decoder, it follows naturally that $C^{Pub} \leq C^{Priv}$. We just need to prove that $C^{Pub} \geq C^{Priv}$ when $D_2 \leq (\sqrt{D_1} + \sigma)^2$. Again, we obtain the lower bound by fixing a (not necessarily optimal) fingerprinting channel and looking for the minimizing collusion attack channel for this specific fingerprinting channel.

Assume that $X_1 = aS + Z_1$, $X_2 = aS + Z_2$, $U_1 = \alpha S + Z_1$, and $U_2 = \alpha S + Z_2$, where $Z_1 \sim \mathcal{N}(0, \sigma_Z^2)$ and $Z_2 \sim \mathcal{N}(0, \sigma_Z^2)$ are independent, and a , α , and σ_Z^2 are to be optimized subject to

$$\mathbb{E}(X_1 - S)^2 = \mathbb{E}(X_2 - S)^2 = (a-1)^2\sigma^2 + \sigma_Z^2 = D_1. \tag{26}$$

Similarly to the steps we take in Section 4.1, we let

$$\begin{aligned}
b &= \frac{\mathbb{E}XY}{\mathbb{E}X^2 + \mathbb{E}X_1X_2}, \\
V &= b^{-1}Y,
\end{aligned}$$

and

$$W = V - X_1 - X_2.$$

Thus

$$\begin{aligned}
Y &= b(X_1 + X_2 + W), \\
\mathbb{E}XW &= \mathbb{E}X_1W = \mathbb{E}X_2W = 0,
\end{aligned}$$

and

$$\mathbb{E}(Y - X_1)^2 = \mathbb{E}(Y - X_2)^2 \leq D_2. \tag{27}$$

Let us denote $\sigma_W^2 = \mathbb{E}W^2$, then

$$\sigma_W^2 \leq b^{-2} [D_2 - (b-1)^2\mathbb{E}X^2 - b^2\mathbb{E}X^2 - 2b(b-1)a^2\sigma^2]. \tag{28}$$

We substitute all the above-defined random variables to the cost function of (9):

$$\begin{aligned}
J &= \frac{1}{2} [I(U_1, U_2; Y) - I(U_1, U_2; S)] \\
&\stackrel{(a)}{=} \frac{1}{2} [I(U_1, U_2; V) - I(U_1, U_2; S)] \\
&\stackrel{(b)}{\geq} \frac{1}{2} [I(U_1, U_2; V^*) - I(U_1, U_2; S)] \\
&\stackrel{(c)}{=} \frac{1}{2} \left[\log \frac{|K_{U_1U_2}| |K_{V^*}|}{|K_{U_1U_2V^*}|} \cdot \frac{|K_{U_1U_2S}|}{|K_{U_1U_2}| |K_S|} \right] \tag{29}
\end{aligned}$$

where (a) follows from the fact that $Y = bV$ is a deterministic function of V ; (b) is obtained because $V^* = X_1 + X_2 + W^*$, where the Gaussian noise $W^* \sim \mathcal{N}(0, \sigma_W^2)$ is independent of X_1 and X_2 , minimizes the mutual information when the noise power is fixed at σ_W^2 ¹⁰; (c) follows from the formula for the mutual information between two Gaussian random vectors

$$I(\mathbf{X}; \mathbf{Y}) = \frac{1}{2} \log \frac{|K_{\mathbf{X}}| |K_{\mathbf{Y}}|}{|K_{\mathbf{XY}}|},$$

where $K_{\mathbf{X}}$ denotes the covariance matrix for the Gaussian random vector $\mathbf{X}$.

The encoder can maximize the right-hand side of (29), the lower bound on J , by selecting α to be

$$\alpha^* = \frac{2a\sigma_Z^2}{2\sigma_Z^2 + \sigma_W^2}$$

so that

$$J \geq \frac{1}{4} \log \left[1 + \frac{2\sigma_Z^2}{\sigma_W^2} \right]. \quad (30)$$

The right-hand side of (30) is the same as the one in (19) for the private fingerprinting case. Consequently, the encoder/decoder applies the same set of optimal parameters as in the private game to maximize the right-hand side of (30), and the colluders do the same to minimize it. Finally, the capacity C^{Pub} is lower bounded by C_G defined in (22). Therefore, $C^{Pub} = C^{Priv}$.

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