ON THE BANDLIMITEDNESS OF THE PLENOPTIC FUNCTION

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ABSTRACT

Image based-rendering (IBR) can be seen as the sampling and reconstruction of the plenoptic function. The question of the minimum sampling rate in IBR can be addressed via spectral analysis of the plenoptic function. We study a model of the scene where bandlimited images are “painted” on surfaces (e.g. of objects or walls). We show that, in general, the plenoptic function is not bandlimited unless the surfaces are flat. We then characterize the spectral decay of the plenoptic function for this model.

1. INTRODUCTION

Existing recording systems use a single camera, and thus provide viewers with a limited and passive viewing experience. Thanks to the continuing improvement in digital technology that now offers low-cost sensors and massive computing power, new systems employing multiple cameras together with sophisticated processing algorithms delivering unprecedented immersive recording and viewing capabilities, are now feasible. Practical synthesis of arbitrary virtual viewpoints from several fixed sensors has already emerged [1, 2].

Synthesizing new views of a scene from a set of acquired views is known as *image-based rendering* (IBR) [3]. IBR can be seen as an application of the sampling theory to the *plenoptic function* [4] that describes the light intensity passing through every viewpoint, in every direction, for all time, and for every wavelength. In this setting, acquired views from the cameras provide discrete samples of the plenoptic function, and the synthesized view is reconstructed from the continuous plenoptic function at a given point.

A convenient way to parameterize the plenoptic function is to use the two-plane parameterization [5, 6], as shown in Figure 1. By restricting the scene in a bounding box, each light ray can be specified by a pair of coordinates (t, u) and (v, w) corresponding to the locations of the camera and the image point, respectively. The image coordinate (v, w) is relatively defined according to the camera position (t, u) .

The value of the plenoptic function $p(t, u, v, w)$ is the light intensity of the intersection of the light ray specified by (t, u, v, w) with the nearest object surface from the camera plane.

Fig. 1. The two-plane parameterization of the plenoptic function. The cameras are placed on the *camera plane* and take pictures of the *image plane*. Each light ray is addressed by a 4-D coordinate (t, u, v, w) .

The first work on sampling analysis for IBR was proposed by Chai et al. [7]. They analyzed the spectral support of the plenoptic function to find an optimal uniform sampling rate of the plenoptic function. In [8], IBR sampling was analyzed for more general cases, including non-Lambertian and occluded scenes. In these works, as for any spectral-based technique, the plenoptic function is assumed to be *bandlimited*.

In this paper, we would like to examine more closely this bandlimited assumption of the plenoptic function. To facilitate this, we will study a simple model where bandlimited images are painted on surfaces (e.g. of objects or walls). Using related mathematical results on time-warped bandlimited signals, we will show that, in general, the plenoptic function for this model is *not* bandlimited unless the surface is flat. We then characterize and approximate the bandwidth expansion due to non-flat surfaces.

2. SCENE MODEL: IMAGES PAINTED ON SURFACES

For simplicity, and as it was done in [7, 8], we consider a 2-D version of the plenoptic function $p(t, v)$ by fixing u and

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w , as shown in Figure 2. This is equivalent to the case where the cameras are placed on a straight line and we consider the same image scan-line from each camera.

We consider a scene model consisted of a surface that is specified by its varying depth $z(t)$ (remember that we consider a slice of the surface, so that it is one-dimensional). An image $f(s)$ is “painted” on this surface, where $s(t)$ is the curvilinear coordinate on the surface. This model represents a micro-scale analysis of the plenoptic function, where locally only one object with its surface is visible.

Fig. 2. Scene model with a functional surface $z(t)$. Coordinates t , v , and z specify the camera position, pixel position, and depth, respectively.

We furthermore fix pixel position v or viewing angle θ as shown in Figure 2. Let $s = s_\theta(t)$ be the curvilinear coordinate on the surface that is intersected by the light ray from the camera position t with viewing angle θ . Then the plenoptic function can be written solely as a function of camera position t as

$$p(t, v_\theta) = f_\theta(s_\theta(t)). \quad (1)$$

where f_θ is the reflected signal of $f(t)$ from the viewing angle θ .

The function $s_\theta(t)$ defines a fundamental mapping that links a light ray with a position on the surface that is “seen” by this light ray. In [8], Zhang and Chen considered a more general notion of this mapping that allows cameras to move in an arbitrary path. In our model, where the cameras are placed in a straight line, $s_\theta(t)$ can be derived from $z(t)$ as follows.

Let $(r, z(r))$ be the t - z coordinate of the intersection point of the surface $z(t)$ with the light ray (t, θ) . Then we have

$$r - \tan(\theta)z(r) = t \quad (2)$$

To write r as a function of t , we need to check if the map $r \mapsto t$ given in (2) is invertible, which means $t_\theta(r) = r - \tan(\theta)z(r)$ is a strictly monotonic function. To satisfy this, we must have

$$\max |z'(r)| < \frac{1}{\tan(\theta_{\max})} \quad (3)$$

Notice that condition (3), which relates the maximum slope of the surface with the maximum viewing angle of the cameras, is exactly the same with the condition requiring no self-occlusion of the surface in the field of view of the cameras.

Given this condition, from (2) we can write r as a function of t as $r = r_\theta(t)$. Using the differential relation $ds = \sqrt{dr^2 + dz^2}$, we finally can derive $s_\theta(t)$ from

$$\begin{aligned} s'_\theta(t) &= \sqrt{[r'_\theta(t)]^2 + [z'(r_\theta(t))r'_\theta(t)]^2} \\ &= \frac{\sqrt{1 + [z'(r_\theta(t))]^2}}{1 - \tan(\theta)z'(r_\theta(t))} \end{aligned} \quad (4)$$

From (4), we see that if the surface is flat, i.e. $z'(t) = \text{const}$, then $s'_\theta(t) = \text{const}$ or $s_\theta(t)$ is an affine function. Conversely, under the no occlusion condition (3), if $s_\theta(t)$ is affine then the surface must be flat. Moreover, (3) ensures that $s_\theta(t)$ is a strictly monotonic function.

3. BANDLIMITED PLENOPTIC FUNCTIONS

As noted in the introduction, to address the sampling problem of the plenoptic function we need to study its spectral support. In this section, we examine the bandlimitedness of the plenoptic function given in (1). For brevity, we will drop the subscript θ and assume it is fixed in this section.

Suppose that the painted signal $f(s)$ is bandlimited. We notice that if the surface in our scene model is flat, then $s(t)$ is affine and the plenoptic function $f(s(t))$ is a uniformly stretched version $f(t)$. Thus, it follows immediately from the shifting and scaling properties of the Fourier transform that $f(s(t))$ is also bandlimited. We are interested to know if there is any other surface that results in bandlimited plenoptic function.

The problem of warped bandlimited functions has been studied in the signal processing literature. In [9], Clark conjectured that when a bandlimited function f is warped by a monotonic function s , the resulting function $f \circ s(t) = f(s(t))$ is also bandlimited if and only if s is affine. In [10], this conjecture was proved for a large class of s , in particular for $s(t)$ that on certain interval is a restriction of an entire function. (An entire function is a function of complex variable that has derivative at each point in the entire finite plane.) Later, in [11], Clark’s conjecture was shown to be false by a counterexample constructed by Y. Meyer. However, the paper also noted that, in general, it is not possible for a non-affine warping function to preserve bandlimitedness in general. Unaware of this line of work, in [12], we made the same conjecture on the preservation of bandlimitedness under warping.

The implication of the above result is that in general, the plenoptic function is *not* bandlimited unless the surface is flat. In the following, we will study the spectral decay of

Fig. 3. Mapping from f to $f \circ s$ due to the pinhole camera projection.

the plenoptic function when the surface deviates from a flat surface, as depicted in Figure 3.

Let $g = f \circ s$ be the plenoptic function as derived in (1). Its Fourier transform is

$$G(\xi) = \int_{-\infty}^{\infty} f(s(t)) e^{-j\xi t} dt. \quad (5)$$

Let $F(\omega)$ be the Fourier transform of $f(s)$. Then,

$$f(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega s} d\omega. \quad (6)$$

Substituting (6) in (5) we obtain

$$\begin{aligned} G(\xi) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left(\int e^{j\omega s(t)} e^{-j\xi t} dt \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) K_s(\xi, \omega) d\omega, \end{aligned} \quad (7)$$

where $K_s(\xi, \omega) = \mathcal{F}_t\{e^{j\omega s(t)}\}$ is the Fourier transform of $e^{j\omega s(t)}$. The kernel function $K_s(\xi, \omega)$ characterizes how the warping function s broadens the spectral of f in the warped function $g = f \circ s$. To see this effect, consider the case when s is an affine function: $s(t) = a + bt$. Then we have

$$K_s(\xi, \omega) = \mathcal{F}_t\{e^{j\omega(a+bt)}\} = 2\pi e^{ja\omega} \delta(\xi - b\omega), \quad (8)$$

which after being substituted in (7) leads to

$$G(\xi) = \frac{e^{ja\xi/b}}{|b|} F\left(\frac{\xi}{b}\right).$$

Thus, for the affine warping $s(t) = a + bt$, if F is bandlimited up to maximum frequency Ω , then G is bandlimited up to maximum frequency $|b|\Omega$.

Now consider the situation where the warping function s deviates from an affine function as

$$s(t) = a + bt + \tilde{s}(t). \quad (9)$$

Then the kernel $K_s(\xi, \omega)$ becomes

$$\begin{aligned} K_s(\xi, \omega) &= \mathcal{F}_t\{e^{j\omega(a+bt)} \cdot e^{j\omega\tilde{s}(t)}\} \\ &= 2\pi e^{ja\omega} \delta(\xi - b\omega) *_{\xi} \mathcal{F}_t\{e^{j\omega\tilde{s}(t)}\}. \end{aligned} \quad (10)$$

Consider a simple case where the deviation $\tilde{s}(t)$ is a periodic function with a single frequency μ , i.e. $\tilde{s}(t) = c \sin(\mu t)$. Using the following expansion

$$e^{jx \sin(\alpha)} = \sum_{n=-\infty}^{\infty} J_n(x) e^{jn\alpha},$$

where $J_n(x)$ is the n -th order Bessel function of the first kind, we have

$$\mathcal{F}_t\{e^{j\omega c \sin(\mu t)}\} = 2\pi \sum_{n=-\infty}^{\infty} J_n(c\omega) \delta(\xi - k\mu).$$

The Bessel functions $J_n(x)$ decay rapidly for large n and can be well approximated as

$$J_n(x) \approx 0 \quad \text{for } |n| > |x| + 1.$$

Thus, when $\tilde{s}(t) = c \sin(\mu t)$, the Fourier transform $\mathcal{F}_t\{e^{j\omega\tilde{s}(t)}\}$ is essentially zero for frequency $|\xi| > \mu + |c\mu\omega|$. Notice that in this case we can write $|c\mu| = \max[\tilde{s}'(t)]$. Based on this approximation, substituting back in (10) and then (7), we see that the essential maximum frequency of G is

$$(|b| + \max[\tilde{s}'(t)])\Omega + \mu.$$

In other words, the derivation by $\tilde{s}(t)$ from an affine warping essentially expands the bandwidth of G by $\mu + \Omega \max[\tilde{s}'(t)]$.

Figure 4 shows an example of the bandwidth expansion due to warping. The warping function is

$$s(t) = \arcsin(t), \quad \text{for } |t| \leq 1,$$

which can be seen as corresponding to the case where the surface is spherical. The “painted” signal is

$$f(s) = \text{sinc}(10s) = \frac{\sin(10\pi s)}{10\pi s},$$

which is bandlimited up to frequency $\Omega = 10\pi$. The Fourier transform of $f \circ s$ in Figure 4(e) shows an expansion from this bandwidth.

4. CONCLUSION

We have studied the bandwidth of the plenoptic function in a simple scene model where a bandlimited signal is painted on a functional surface. We show that in general the plenoptic function for this model is *not* bandlimited unless the surface is flat. We also show that for this model the spectrum of the plenoptic function is negligible beyond a frequency that depends of the maximum frequency of the painted signal, and the derivative of the warped function.

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5. REFERENCES

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Fig. 4. Example showing the bandwidth expansion due to warping.