

# Joint Source–Channel Decoding for Transmitting Correlated Sources over Broadcast Networks

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**Abstract**—We consider a set of  $S$  independent encoders that must transmit a set of correlated sources through a network of noisy, independent, broadcast channels to  $T$  receivers. For the general problem of sending correlated sources through broadcast networks, it is known that the source–channel separation theorem breaks down and the achievable rate region as well as the proper method of coding are unknown.

For our scenario, however, we not only establish the optimal rate region, but we show that a type of source–channel separation is possible at the transmitter, provided joint source–channel decoding is used at the receiver. Furthermore, we show that while joint source–channel encoding is unnecessary, not using joint source–channel decoding is suboptimal. Finally, when the optimal input distribution from transmitter  $i$  to receiver  $j$  is independent of  $j$ , our result has a max-flow/min-cut interpretation. Specifically, in this case our result implies that if it is possible to send the sources to each receiver separately while ignoring the others, then it is possible to send to all receivers simultaneously.

## I. INTRODUCTION

While many point-to-point communication problems are relatively well understood, the more general problem of how to transmit correlated sources to multiple receivers is much harder to analyze. Two of the key features that complicate network information theory problems include the lack of source–channel separation and the conflicting goals that arise when multiple destinations receive the same transmission (broadcast) or when multiple transmitters send to the same destination (multiple access).

In order to understand how these issues effect communication system design, we consider the problem of sending  $S$  correlated sources through a noisy broadcast network with  $T$  receivers as illustrated in Figure 1. Specifically, we imagine that transmitter  $i$  observes a source  $U_i$  which consists of a set of  $n$  independent and identically distributed samples,  $U_i[1], U_i[2], \dots, U_i[n]$ . The sources are correlated (across the transmitters) in the sense that the joint distribution (across  $i$ ) is

$$P(U_1, U_2, \dots, U_S) = \prod_{m=1}^n P(U_1[m], U_2[m], \dots, U_S[m]). \quad (1)$$

The channels are independent broadcast channels in the sense that if each transmitter chooses a block of channel inputs  $X_i[1], X_i[2], \dots, X_i[n]$  then receiver  $j$  observes the  $i$  channels without interference to receive the outputs  $Y_{i,j}[m]$  according

to the channel law

$$P(Y_{1,1} \dots Y_{S,T} | X_1 \dots X_S) = \prod_{m=1}^n \prod_{i=1}^S \prod_{j=1}^T P_{i,j}(Y_{i,j}[m] | X_i[m]) \quad (2)$$

Tuncel considered the related problem of using joint source–channel coding for broadcasting correlated sources [1] while Cover, El Gamal, and Salehi [2] considered a similar problem for multiple access networks. Specifically, in the latter model, there is only a single receiver (*i.e.*,  $T = 1$ ), but the channel is a multiple access channel and does not factor as in (2). As a result, [2] shows that the source–channel separation theorem for point-to-point communication [3, Sec. 14.10] breaks down and a joint source–channel coding strategy is required.<sup>1</sup>

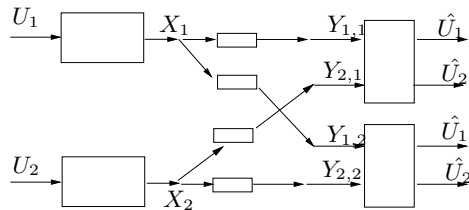


Fig. 1. Transmitting Correlated Sources over a Broadcast Network. Two correlated sources  $U_1$  and  $U_2$  are observed by independent transmitters and both sources must be communicated to each receiver.

In contrast, as in the case of [4], a key feature of our network model is that each receiver observes the output of the channel from each transmitter independently and so there is no multiple access interference. Consequently, effects such as coherent or incoherent interference, beam-forming, or transmitter cooperation are not possible. Note, however, that there is a type of “broadcast interference” in the sense that transmitter  $i$  cannot choose to send independent messages to each receiver. Instead, transmitter  $i$  must choose a channel input that simultaneously conveys the desired message to all receivers. Similarly, there is also a kind of “distribution interference” in the sense that transmitter  $i$  cannot choose a different input distribution for each channel. For example, if  $Y_{1,1}$  is the result of passing  $X_1$  through an additive white Gaussian noise channel while  $Y_{1,2}$  is the result of passing  $X_1$  through an additive exponential noise channel, then it is not possible to simultaneously transmit at the point-to-point capacity of the individual channels.

One of our key results is that while separate source and channel coding is sub-optimal for the scenario in this paper, it

<sup>1</sup>Specifically, [2] shows that both joint source–channel encoding and joint source–channel decoding are required in general.

is possible to achieve optimal performance with joint source–channel decoding but separate encoding. Specifically, it suffices to consider modular encoders that first perform source coding (by mapping the source into bits) and then perform channel coding (by mapping bits into a channel input) where the operations can be designed separately.

This result has two implications. First, instead of thinking of joint vs. separate coding we should think of joint vs. separate encoding and joint vs. separate decoding. Second, “broadcast interference” causes a loss of separation in source–channel decoding but no loss of separation in encoding. Third, “multiple access” interference (*e.g.*, in the Cover, El Gamal, Salehi scenario) causes a loss of separation in source–channel encoding (and potentially decoding as well).

## II. STATEMENT OF RESULTS

Let  $C_{i,j}$  denote the point-to-point capacity of the channel connecting source  $i$  to receiver  $j$ :

$$C_{i,j} \triangleq \max_{P(x_i)} I(X_i; Y_{i,j}). \quad (3)$$

Specifically,  $C_{i,j}$  is the maximum rate of information that can be sent from transmitter  $i$  to receiver  $j$ , if all other receivers are ignored. Thus, if we only cared about receiver  $j$  it is trivial to determine whether all the sources could be successfully communicated to the receiver with no bandwidth expansion/compression:<sup>2</sup>

*Proposition 1:* If we only consider receiver  $j$ , the sources can be successfully communicated to receiver  $j$  if and only if

$$(C_{1,j}, C_{2,j}, \dots, C_{S,j}) \in \mathcal{R}_{\text{SW}}(U_1, U_2, \dots, U_S) \quad (4)$$

where  $\mathcal{R}_{\text{SW}}(U_1, U_2, \dots, U_S)$  is the Slepian-Wolf [5] rate region for the sources  $U_1, U_2, \dots, U_S$ .

While (4) is sufficient if we only want to communicate to receiver  $j$ , it is not sufficient if we wish to communicate to all receivers simultaneously, in a source–channel separated fashion. Specifically, even if (4) is satisfied for each  $j$ , it may be impossible to successfully communicate the sources to all the receivers with separate source and channel decoding.

For example, suppose each channel is a binary symmetric channel (BSC) where  $C_{1,1} = 0.9, C_{1,2} = 0.6, C_{2,1} = 0.6, C_{2,2} = 0.9$  in Figure 1 and suppose we would like to communicate the correlated sources  $U_1$  and  $U_2$  where  $H(U_1) = H(U_2) = 1, H(U_1, U_2) = 1.5$ . Then from the cut-set bounds [3], if we were to perform complete source-channel separation at the encoders and decoders, then message 1 can be decoded by receiver 2 only if  $R_1 \leq 0.6$ . Similarly, message 2 can be decoded by receiver 1 only if  $R_2 \leq 0.6$ . This implies that  $R_1 + R_2 \leq 1.2 < H(U, V)$ . So performing Slepian-Wolf encoding and decoding in a modular fashion with channel

<sup>2</sup>With no bandwidth expansion/compression, if  $n$  source samples are observed for each  $U_i$ , then exactly  $n$  channel samples are used. In general, one could analyze the ratio of source samples to channel samples required. While this ratio is interesting in its own right, we do not consider it here since our goal is to prove the qualitative results on separability of source–channel encoding and max-flows/min-cuts.

coding does not suffice even though (4) is satisfied for all  $j$ .

Any scheme based on Slepian-Wolf source coding followed by channel encoding for the above example would require transmitting across some of the BSCs at rates greater than their capacities. Nevertheless, the following theorem, which is our main result, implies that the use of a certain joint source–channel decoding procedure does permit the reliable transmission of the sources to each of the receivers.

*Theorem 1:* The sources  $U_1, \dots, U_S$  can be reliably communicated to all receivers simultaneously using separate source and channel encoding and joint source–channel decoding if there exist random variables  $X_1, \dots, X_S$  satisfying

$$H(U_{\mathcal{A}}|U_{\mathcal{A}^c}) < \sum_{i \in \mathcal{A}} I(X_i; Y_{i,j}) \quad \forall j \in \mathcal{T} \quad (5)$$

for all  $\mathcal{A} \subseteq \mathcal{S}$ ,<sup>3</sup> where  $U_{\mathcal{A}} = \{U_i\}_{i \in \mathcal{A}}$ . Conversely, if the sources  $U_1, \dots, U_S$  can be reliably communicated to all receivers simultaneously, then there exist random variables  $X_1, \dots, X_S$  satisfying

$$H(U_{\mathcal{A}}|U_{\mathcal{A}^c}) \leq \sum_{i \in \mathcal{A}} I(X_i; Y_{i,j}) \quad \forall j \in \mathcal{T} \quad (6)$$

for all  $\mathcal{A} \subseteq \mathcal{S}$ , where  $U_{\mathcal{A}} = \{U_i\}_{i \in \mathcal{A}}$ .

Consider the case where for each  $i$ , there is an optimal input distribution  $P(x_i^*)$  that simultaneously achieves the point-to-point capacity from transmitter  $i$  to receiver  $j$ :

$$\forall i, \exists P(x_i^*) \text{ such that } \forall j, I(X_i^*; Y_{i,j}) = C_{i,j}. \quad (7)$$

When this condition is satisfied (*e.g.*, when each channel from  $i$  to  $j$  is an additive white Gaussian noise channel), Theorem 1 has a max-flow/min-cut interpretation. Specifically, if we consider a cut-set separating receiver  $j$  from all the transmitters we can easily determine whether receiver  $j$  can decode all the sources using capacity arguments and the Slepian-Wolf theorem via Proposition 1. Evidently when (7) is true, Theorem 1 says that we can determine whether a given communication problem is feasible (*i.e.*, whether all the receivers can successfully decode all the sources) simply by considering the minimum cut. It seems that in this scenario, information acts like a fluid in the sense that analyzing flows is sufficient and worrying about “broadcast interference” is not required.

## III. PROOF SKETCH

### A. Achievability with Separate Source-Channel Encoding and Joint Source-Channel Decoding

Here we perform separate separate source and channel coding at the encoders but perform joint source channel decoding (see Figure 2).

<sup>3</sup>We use the notation  $\mathcal{S}$  to denote the set  $\{1, 2, \dots, S\}$  and  $\mathcal{T}$  to denote the set  $\{1, 2, \dots, T\}$ .

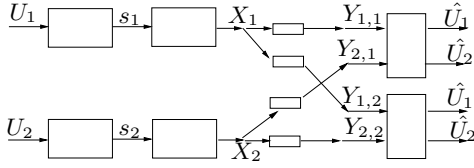


Fig. 2. Channel Figure with Separate Source-Channel Encoding and Joint Source-Channel Decoding

1) *Encoder*: For each  $i \in \mathcal{S}$ : randomly bin source sequences  $u_i^n$  at rate

$$R_i = \max_{j \in \mathcal{T}} I(X_i, Y_{i,j}) \quad (8)$$

and generate  $2^{nR_i}$  codewords  $x_i^n(s)$  ( $s = 1, \dots, 2^{nR_i}$ ) with symbols independently and identically distributed according to  $P_{X_i}(x_i)$ . So a source sequence  $u_i^n$  is first mapped to a bin index  $s_i = f_i(u_i^n)$ , and then  $s_i$  is mapped to the channel codeword  $x_i^n(s_i)$ .

2) *Decoder*: At the  $j$ th joint source-channel decoder, create a list  $\{(\tilde{s}_1, \dots, \tilde{s}_S)\}$  of all bin tuples where each corresponding  $x_i^n(\tilde{s}_i)$  is jointly typical with the received  $Y_{i,j}^n$ . Decode to a jointly typical set of source sequences  $(\tilde{u}_1^n, \dots, \tilde{u}_S^n)$  lying in  $\{(\tilde{s}_1, \dots, \tilde{s}_S)\}$ .

We shall use the upper case symbols  $U_i^n, X^n(s)$ , and  $S_i = F_i(u^n)$  to denote the random values of the source sequence, channel codewords, and bin indices, respectively, with distributions induced by the source distribution and the above random encoder selection.

3) *Error Probability Analysis*: We denote by  $A_\epsilon^n$  the typical set. The error probability  $P_e$  can be bounded by

$$P_e \leq \sum_{j \in \mathcal{T}} [P(E_{j,0}) + \sum_{A \subseteq \mathcal{S}} P(E_{j,A})] \quad (9)$$

where:

- $E_{j,0} \triangleq \{U_S^n \notin A_\epsilon^n\} \cup \bigcup_{i=1}^S \{(X_i^n(F_i(U_i^n)), Y_{i,j}^n) \notin A_\epsilon^n\}$ . The probability of this event, averaged over the random binning and channel codes, tends to 0 by the Asymptotic Equipartition Property [3].
- $E_{j,A} \triangleq \left\{ \exists u_{\mathcal{A}}^n : (u_{\mathcal{A}}^n \neq U_{\mathcal{A}}^n)_{i \in \mathcal{A}}, (u_{\mathcal{A}^c}^n, U_{\mathcal{A}^c}^n) \in A_\epsilon^n, (X_i^n(F_i(u_i^n)), Y_{i,j}^n) \in A_\epsilon^n \forall i \in \mathcal{A} \right\}$ .

We now bound  $\bar{P}(E_{j,A})$  which is  $P(E_{j,A})$  averaged over the random binning and channel codes. Note that for each  $i \in \mathcal{A}$ , by (8), there are about  $2^{n[R_i - I(X_i; Y_{i,j})]}$  bin indices  $\{s\}$  such that  $(X_i^n(s), Y_{i,j}^n) \in A_\epsilon^n$ . For any  $s$  and  $u^n$ , the probability that  $F_i(u^n) = s$  is  $2^{-nR_i}$ . Since the random bin index  $F_i(u^n)$  of a sequence  $u^n \neq U_i^n$  is independent of  $Y_{i,j}^n$  and of all the codewords, the probability that  $F_i(u^n)$  satisfies  $(X_i^n(F_i(u^n)), Y_{i,j}^n) \in A_\epsilon^n$  is approximately  $2^{n[R_i - I(X_i; Y_{i,j})]} 2^{-nR_i} = 2^{-nI(X_i; Y_{i,j})}$ . By the independence of the binning across encoders, the probability that this holds for any set of  $u_i^n, i \in \mathcal{A}$  with  $u_i^n \neq U_i^n$ , is about

$$2^{-n \sum_{i \in \mathcal{A}} I(X_i; Y_{i,j})}. \quad (10)$$

Therefore

$$\begin{aligned} \bar{P}(E_{j,A}) &= P\left( \bigcup_{\substack{u_{\mathcal{A}}^n : (u_{\mathcal{A}}^n, U_{\mathcal{A}^c}^n) \in A_\epsilon^n \\ (u_i^n \neq U_i^n)_{i \in \mathcal{A}}}} \bigcap_{i \in \mathcal{A}} \{(X_i^n(F_i(u_i^n)), Y_{i,j}^n) \in A_\epsilon^n\} \right) \\ &\leq 2^{nH(U_{\mathcal{A}}|U_{\mathcal{A}^c})} 2^{-n \sum_{i \in \mathcal{A}} I(X_i; Y_{i,j})} \quad (11) \\ &\rightarrow 0 \quad (12) \end{aligned}$$

where (11) follows from a union bound argument conditioned on  $U_{\mathcal{S}}^n$ , the independence of the binning and the channel codewords from the source, the probability (10), and the fact that with high probability

$$|\{u_{\mathcal{A}}^n : (u_{\mathcal{A}}^n, U_{\mathcal{A}^c}^n) \in A_\epsilon^n\}| \approx 2^{nH(U_{\mathcal{A}}|U_{\mathcal{A}^c})}.$$

Finally, (12) is implied by (5).

## B. Converse

Let us denote the error probability at receiver  $j \in \mathcal{T}$  as  $P_e^j$ . For each  $\mathcal{A} \subseteq \mathcal{S}$ , define

$$\begin{aligned} Y_{\mathcal{A},j}^n &\triangleq \{Y_{i,j}^n\}_{i \in \mathcal{A}} \\ X_{\mathcal{A}}^n &\triangleq \{X_i^n\}_{i \in \mathcal{A}}. \end{aligned}$$

WLOG, assume that  $\mathcal{A} = \{1, \dots, |\mathcal{A}|\}$ . Fano's inequality tells us that if  $P_e^j \rightarrow 0$  as  $n \rightarrow \infty$ , then:

$$H(U_{\mathcal{A}}^n | Y_{\mathcal{A},j}^n, U_{\mathcal{A}^c}^n) \leq n\epsilon_n \text{ where } \epsilon_n \rightarrow 0 \quad (13)$$

Thus

$$\begin{aligned} H(U_{\mathcal{A}}^n | U_{\mathcal{A}^c}^n) &= H(U_{\mathcal{A}}^n | U_{\mathcal{A}^c}^n, X_{\mathcal{A}}^n) \\ &= I(U_{\mathcal{A}}^n; Y_{\mathcal{A},j}^n | U_{\mathcal{A}^c}^n, X_{\mathcal{A}}^n) \\ &\quad + H(U_{\mathcal{A}}^n | Y_{\mathcal{A},j}^n, U_{\mathcal{A}^c}^n, X_{\mathcal{A}}^n) \\ &\leq I(U_{\mathcal{A}}^n; Y_{\mathcal{A},j}^n | U_{\mathcal{A}^c}^n, X_{\mathcal{A}}^n) + n\epsilon_n \quad (14) \\ &\leq I(X_{\mathcal{A}}^n; Y_{\mathcal{A},j}^n | U_{\mathcal{A}^c}^n, X_{\mathcal{A}}^n) + n\epsilon_n \\ &= H(Y_{\mathcal{A},j}^n | U_{\mathcal{A}^c}^n, X_{\mathcal{A}}^n) \\ &\quad - H(Y_{\mathcal{A},j}^n | X_{\mathcal{A}}^n, U_{\mathcal{A}^c}^n) + n\epsilon_n \\ &\leq H(Y_{\mathcal{A},j}^n) - H(Y_{\mathcal{A},j}^n | X_{\mathcal{A}}^n, U_{\mathcal{A}^c}^n) + n\epsilon_n \\ &= H(Y_{\mathcal{A},j}^n) \\ &\quad - \sum_{a \in \mathcal{A}} H(Y_{a,j}^n | \{Y_{a',j}^n\}_{\{a' < a\}}, X_{\mathcal{A}}^n, U_{\mathcal{A}^c}^n) + n\epsilon_n \\ &= H(Y_{\mathcal{A},j}^n) - \sum_{a \in \mathcal{A}} H(Y_{a,j}^n | X_a^n) + n\epsilon_n \quad (15) \\ &\leq \sum_{a \in \mathcal{A}} H(Y_{a,j}^n) - H(Y_{a,j}^n | X_a^n) + n\epsilon_n \\ &\leq \sum_{a \in \mathcal{A}} \sum_{t=1}^n I(X_a[t]; Y_{a,j}[t]) + n\epsilon_n \quad (16) \\ &\leq n \sum_{a \in \mathcal{A}} I(X_a^*; Y_{a,j}) + n\epsilon_n \quad (17) \end{aligned}$$

where (14) is due to (13), (15) follows because the channels are independent, (16) follows because the channels are memoryless and because conditioning reduces entropy, and (17)

follows by defining

$$P(X_a^*) = \frac{1}{n} \sum_{t=1}^n P(X_a[t])$$

and noting the concavity of mutual information.

#### IV. CONCLUDING REMARKS

To summarize, we illustrated that for broadcast networks, performing source-channel separation for encoding and decoding is in general suboptimal. Next, we showed that separate encoding and joint source-channel decoding attains optimal performance. Thus it appears that the need for joint-source channel *encoding* in multiple access networks essentially comes from the need to allow for the correlation in the sources to be preserved to aid in the signaling across the multiple access channels *e.g.*, for beamforming (or other types of coherent transmission). What we have shown is that in broadcast networks where such signaling provides no benefit, something still can be gained by doing separate source-channel encoding and joint-source channel decoding.

This may have practical relevance - for example, in sensor networks with fusion centers, processing power is usually not nearly as constrained of a resource for the fusion center as is the case for the individual sensors. Consequently making the signaling methods for the individual sensor nodes to require the least possible cooperation and computation is generally desirable. Note that these properties fit well with the achievability methods we proposed here, and they still attain asymptotically optimal performance.

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